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LETTER TO THE EDITOR

New examples of solvable non-uniform spin lattices

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Abstract. We propose a three-parametric extension of integrable non-uniform spin lattices in 1D. In our model, the spins are located at the equilibrium positions of particles interacting via the potential $(\sinh(r))^{-2}$, the particles being confined by some external field. The Lax representation, conserved current and special set of eigenvectors are presented in explicit form.

In recent years, there has been renewed interest to the problem of finding exact solutions to many-body lattice systems [1–6]. The formal simplicity of the spectrum of the integrable Haldane–Shastry $S = \frac{1}{2}$ spin chain [1, 2] with the interaction between spins being proportional to the inverse square of their chord distance stimulated the search for and investigation of its various extensions and generalizations [2–6] connected with integrable *classical* Calogero–Sutherland particle systems on a continuous line. The family of the latter is rather rich, and in the general hyperbolic case it includes interaction with a three-parametric external field [7]. Despite the above correspondence having been investigated in detail for Calogero systems in a field with a harmonic oscillator [3] and hyperbolic Sutherland systems in a field with a Morse potential [5], the general situation has not so far been clear.

In this letter, we shall give the arguments supporting the general statement about one-to-one correspondence between integrable classical Sutherland systems in external fields and solvable spin chains. Moreover, we are able to present some eigenvectors of these inhomogeneous chains in explicit form for the case of $S = \frac{1}{2}$.

It is generally believed that all the family of solvable spin chains of that type might be viewed as the static ($\lambda \to \infty$) limit of the systems containing particles with spatial and internal degrees of freedom with the Hamiltonian

$$\mathcal{H}^{(N)} = \sum_{j=1}^{N}[p_j^2/2 + W(x_j)] + \sum_{j<k}^{N}\lambda(\lambda + P_{jk})V(x_j - x_k) \tag{1}$$

where the $\{x_j, p_j\}$ are particle coordinates and momenta, the $\{P_{jk}\}$ act in the internal space as permutations and

$$V(x) = x^{-2} \qquad \text{or} \qquad [\sinh(x)]^{-2}. \tag{2}$$

The integrability of the systems described by (1) without internal degrees of freedom has been established in [7] in the latter case of (2) for the following choice of the potential of an external field:

$$W(x) = \alpha^2\cosh(4x) + 2\beta\cosh(2x) + 2\gamma\sinh(2x). \tag{3}$$

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In the rational Calogero case of (2), the integrability of (1) has been proved in [3] for the harmonic external potential by constructing a special set of operators commuting with (1). Later, it was found that this generalized Calogero system has a rather simple equidistant spectrum and posesses explicit Yangian symmetry via a special form of the Lax representation [8].

In the general hyperbolic case, the Hamiltonian equations of motion can also be written in some Lax form. It turns out, however, that this form is inappropriate for obtaining conserved currents for the model. Moreover, the Hamiltonian cannot be found among the invariants of the L matrix which do not commute with it. For this reason, we shall restrict ourselves to the non-uniform lattice spin model described by the Hamiltonian

$$H = \sum_{j\leqslant k}^{N} h_{jk} P_{jk} \tag{4}$$

where

$$h_{jk} = \sinh^{-2}(x_j - x_k) \tag{5}$$

and $\{x_j\}$ are the coordinates of classical particles at equilibrium satisfying the equations

$$-2\sum_{k\neq j} h_{jk} c_{jk} + W'(x_j) = 0 \tag{6}$$

where $c_{jk} = \coth(x_j - x_k)$. To construct the Lax pair for this system, consider the following ansatz resembling that found in [7, 9]:

$$L = \begin{pmatrix} L_0 & \psi + \rho \\ -\psi - \rho & -L_0 \end{pmatrix} \qquad M = \begin{pmatrix} M_0 + m & \phi \\ \phi & M_0 + m \end{pmatrix}$$

where L_0 and M_0 constitute the standard Lax pair for the Sutherland N-particle system [9]:

$$(L_0)_{jk} = (1 - \delta_{jk}) c_{jk} P_{jk} \qquad (M_0)_{jk} = (1 - \delta_{jk}) h_{jk} P_{jk} - \delta_{jk} \sum_{s\neq j}^{N} h_{js} P_{js}$$

and ψ, ϕ, ρ and m are $(N \times N)$ matrices with entries

$$(\psi)_{jk} = \xi(z_j)\delta_{jk} \qquad (\phi)_{jk} = \varphi(z_j)\delta_{jk} \qquad (m)_{jk} = \mu(z_j)\delta_{jk} \qquad (\rho)_{jk} = P_{jk}(1 - \delta_{jk})$$

where $z_j = \exp(2x_j)$. One finds that the Lax relation $[H, L] = [L, M]$ is equivalent to the overdetermined set of functional equations

$$c_{jk}[\mu(z_j) - \mu(z_k)] + [\varphi(z_j) + \varphi(z_k)] = 0$$

$$c_{jk}[\varphi(z_j) + \varphi(z_k)] + h_{jk}[\xi(z_j) - \xi(z_k)] + \mu(z_j) - \mu(z_k) = 0.$$

Starting from the more restrictive first equation, one obtains the general solution of this set in the form

$$\mu(z) = \mu_1 z + \mu_2 z^{-1} \qquad \varphi(z) = -\mu_1 z + \mu_2 z^{-1} \qquad \xi(z) = \mu_1 z + \mu_2 z^{-1} + \gamma.$$

The corresponding potential of the external field reads

$$W(z) = 2\left[\mu_1^2 z^2 + \mu_2^2 z^{-2} + (2\gamma - 1/2)(\mu_1 z + \mu_2 z^{-1})\right].$$

It depends on three arbitrary parameters as in (3). However, one can see from the form of the Lax pair that the matrix M obeys the condition $\sum_{j=1}^{2N} M_{jk} = 0$, which guarantees its usefulness for constructing integrals of motion [8] only for the special one-parametric case of the Morse potential [5]. In other cases, as mentioned above, the existence of the Lax representation does not allow one to find the explicit form of the conserved quantities.

It is quite natural to assume that extra constants of motion for the model (4), (5) are given in terms of polynomials in the permutations $\{P_{jk}\}$, as occurs for all the lattice spin models known at this time [3–6]. We have found, by modification of the traces of the L matrix, that the minimal degree of these polynomials is 3. Moreover, the operator

$$\mathcal{I} = \sum_{j\neq k\neq l\neq m}^{N} c_{jk}c_{kl}P_{jklm} - \frac{1}{2}\sum_{j\neq k\neq l}^{N}(c_{jl}-c_{kl})^2 P_{jk} + \sum_{j\neq k}^{N}(F(x_j)+F(x_k))P_{jk}$$

where $P_{jklm} = P_{jk}P_{kl}P_{lm}$ is the cyclic permutation, commutes with H if F obeys the relation

$$g(x_j, x_k) + g(x_k, x_l) + g(x_l, x_j) = 0$$

with

$$g(x_j, x_k) = 2h_{jk}(F(x_j) - F(x_k)) + c_{jk}(W'(x_j) + W'(x_k)).$$

As a consequence, one finds $g(x_j, x_k) = G(x_j) - G(x_k)$, $F(x) = -W(x)$ and the functional equation for the potential $W(x)$:

$$c_{jk}(W'(x_j) + W'(x_k)) - 2h_{jk}(W(x_j) - W(x_k)) = G(x_j) - G(x_k).$$

Its general solution contains three arbitrary parameters and reproduces the potential (3) which is already known from the integrability condition for classical particle systems and the existence of the Lax representation for the lattice Hamiltonian (4). To construct the explicit eigenvectors of our spin Hamiltonian, one needs more information about the equilibrium part of its classical counterpart

$$\mathcal{H} = \sum_{j<k}^{N}\frac{z_j z_k}{(z_j - z_k)^2} + \frac{1}{4}\sum_{j=1}^{N}\left[\frac{\alpha^2}{2}(z_j^2 + z_j^{-2}) + (\beta+\gamma)z_j + (\beta-\gamma)z_j^{-1}\right] \tag{7}$$

(recall that $z_j = \exp(2x_j)$) and the corresponding equilibrium equations

$$-\sum_{k\neq j}^{N}\frac{z_k(z_j + z_k)}{(z_j - z_k)^3} + \frac{1}{4}[\alpha^2(z_j - z_j^{-3}) + \beta + \gamma - (\beta-\gamma)z_j^{-2}] = 0. \tag{8}$$

The latter are in general highly nonlinear and the properties of their solutions are far from obvious. To investigate some of them, the trick used in [10–12] might be applied.

Following Ahmed [11], let us denote as $P_N(z)$ a polynomial constructed from the solution $\{z_j\}$ of the equilibrium equations

$$P_N(z) = \prod_{j=1}^{N}(z - z_j). \tag{9}$$

Note that for each j one can define the function $F_j(z) = z(z+z_j)(z-z_j)^{-3}\mathrm{d}\log P_N(z)/\mathrm{d}z$ which is proportional to z^{-2} as $z\to\infty$ and has simple poles at $z = z_k$ $(k\neq j)$ with residues $\mathrm{res}\, F_j(z)|_{z=z_k} = -z_k(z_j+z_k)(z_j-z_k)^{-3}$. Then the equilibrium equations (8) can be recast in the form

$$\begin{aligned}\mathrm{res}\, F_j(z)|_{z=z_j} &= 2a_{1j} + z_j(4a_{2j} - 3a_{1j}^2) + z_j^2(a_{3j} + a_{1j}^3 - 2a_{1j}a_{2j})\\ &= \alpha^2(z_j - z_j^{-3}) + \beta + \gamma - (\beta-\gamma)z_j^{-2}\end{aligned} \tag{10}$$

where $a_{\lambda j} = [P_N'(z_j)]^{-1}(\mathrm{d}/\mathrm{d}z)^{\lambda+1}P_N(z)|_{z=z_j}$. Let us now suppose that $P_N(z)$ obeys the second-order differential equation

$$z^2 P_N''(z) + w_1(z)P_N'(z) + w_2(z)P_N(z) = 0 \tag{11}$$

where $w_{1,2}(z)$ are some polynomials in z. By using the equality $P_N(z_j) = 0$, one finds upon consecutive differentiation of (11) that res $F_j(z)|_{z=z_j}$ is expressed through $w_{1,2}$, and that equation (10) is equivalent to the relation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[w_2 + \frac{1}{4}(\alpha^2(z^2 + z^{-2}) - z^{-2}w_1^2) + \frac{1}{2}w_1' + 2(z(\beta+\gamma) + (\beta-\gamma)z^{-1})\right] = 0.$$

It is satisfied by $w_1(z) = -\alpha(z^2 - 1) + (4\alpha^{-1}\beta - \gamma_1)z$, $w_2(z) = (\alpha - 4\beta)z + \varepsilon_N$, where $\gamma_1 = 4\alpha^{-1}\gamma$. One of the solutions to (11) is an Nth-degree polynomial if the parameters α and β are correlated:

$$\beta = -\frac{N-1}{4}\alpha. \tag{12}$$

Taking this constraint into account, one can finally write equation (11) in the form

$$z^2 P_N''(z) - [\alpha(z^2 - 1) + (\gamma_1 + N - 1)z]P_N'(z) + [\alpha N z + \varepsilon_N]P_N(z) = 0. \tag{13}$$

It has two irregular points located at $z = 0, \infty$ and thus cannot be reduced to the equation of the Bessel or hypergeometric type. The substitution $P_N(z) = \sum_{l=0}^{N-1} d_l z^l + z^N$ yields the recurrence relation for the coefficients $\{d_l\}$

$$\alpha d_{l-1}(N - l + 1) + d_l[\varepsilon_N + l(l - \gamma_1 - N)] + \alpha(l+1)d_{l+1} = 0 \qquad l = 0, \ldots, N \tag{14}$$

which has to be treated under the conditions

$$d_{-1} = 0 \qquad d_N = 1 \qquad d_{N+1} = 0. \tag{15}$$

This results in an Nth-order algebraic equation in the parameter ε_N, the proper solution being determined by the positivity of all the roots of $P_N(z)$. It is unique since the system of particles with repulsive interaction has only one equilibrium position being confined by the external field (3). It is worth noting that (13)–(15) allow one to express various sums $S_\delta = \sum_{j=1}^{N} z_j^\delta$ of the roots of (8) in terms of α, γ_1 and ε_N; in particular, some of these are given by

$$S_{-2} = N + \varepsilon_N\alpha^{-2}(\gamma_1 + N - 1) \qquad S_{-1} = \alpha^{-1}\varepsilon_N$$

$$S_1 = \alpha^{-1}(\varepsilon_N - N\gamma_1) \qquad S_2 = N - \alpha^{-2}(\varepsilon_N - N\gamma_1)(\gamma_1 - N + 1).$$

The energy (7) of the classical equilibrium configuration does not depend on ε_N:

$$E_{\mathrm{cl}} = -\frac{N}{2}\left[\frac{N^2 - 1}{3} + \gamma_1^2 - 2\alpha^2\right].$$

Having prepared the solutions to the equilibrium equations in a suitable form, let us now consider the quantum spin chain with the Hamiltonian (4) under constraint (12) and construct its eigenvectors by the action of the spin-flip operators $\{\sigma_j^-\}$ on one of the ferromagnetic vacua $|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$. The eigenvectors in sectors $S_z = N/2 - M$ can be written as

$$|\phi_M\rangle = \sum_{j_1 \neq j_2 \cdots \neq j_M}^{N} \phi_{j_1\cdots j_M} \prod_{\alpha=1}^{M} \sigma_{j_\alpha}^- |0\rangle. \tag{16}$$

As for the amplitudes of the excitations in the one-magnon sector of the model, we use the ansatz

$$\phi_j = \frac{Q(z_j)}{P_N'(z_j)}$$

where $Q(z)$ is some polynomial. Using the properties of $\{z_j\}$, we find that the eigenproblem $\mathcal{H}|\phi_1\rangle = \epsilon_1|\phi_1\rangle$ is equivalent to the differential equation

$$z^2 Q'' - Q'[\alpha(z^2-1) + (\gamma_1 + N - 3)z] + Q[\alpha(N-2)z + \varepsilon_N + 2\epsilon_1 - (\gamma_1 + N - 1)] = 0. \tag{17}$$

An analysis similar to that for (13) shows that there are $N-1$ polynomial solutions to (17) which correspond to different choices of the eigenenergy ϵ_1. Together with the trivial solution to the eigenproblem, ϕ_j = constant, they give a complete set of one-magnon excitations.

In M-magnon sectors with $M \leqslant N/2$, we work with the ansatz

$$\phi_{j_1\cdots j_M} = \frac{\prod_{\lambda>\mu}^{M}(z_{j_\lambda} - z_{j_\mu})^2}{\prod_{\mu=1}^{M} P_N'(z_{j_\mu})} Q(z_{j_1}, \ldots, z_{j_M})$$

where $Q(z_1, \ldots, z_M)$ is some symmetric polynomial in $\{z\}$ of degree $D \leqslant N - 2M + 1$, and find that the eigenequation

$$\sum_{s=1}^{M}\sum_{k\neq j_s}^{N} h_{j_s k}[\phi_{j_1\cdots j_{s-1}kj_{s+1}\cdots j_M} - \phi_{j_1\cdots j_M}] + \left(\sum_{s\neq k}^{M} h_{j_s j_k} + \epsilon_M\right)\phi_{j_1\cdots j_M} = 0$$

can be cast in the form of a second-order partial differential equation for Q:

$$\sum_{j=1}^{M}\left\{z_j^2\frac{\partial^2}{\partial z_j^2} - [\alpha(z_j^2-1) + (\gamma_1 + N - 3)z_j]\frac{\partial}{\partial z_j}\right\} + 2\sum_{j\neq k}^{M}\frac{z_j^2\partial/\partial z_j - z_k^2\partial/\partial z_k}{z_j - z_k}Q$$
$$+\left\{M[(M-1)(4M+1)/3 - M(\gamma_1 + N - 1) + \varepsilon_N]\right.$$
$$\left.+\alpha(N-2M)\sum_{k=1}^{M} z_k - 2\epsilon_M\right\}Q = 0. \tag{18}$$

For even N, an obvious solution to (18) in the sector with $S = S_z = 0$ ($M = N/2$) is Q = constant, which gives an eigenenergy

$$\epsilon_{N/2} = 1/2\{M[(M-1)(4M+1)/3 - M(\gamma_1 + N - 1) + \varepsilon_N]\}.$$

For non-uniform lattices with $N \leqslant 8$ under various choices of parameters in (3) constrained by (12) we have verified numerically that this is the exact ground-state energy of the antiferromagnetic spin chain with the Hamiltonian (4). However, we did not succeed in obtaining analytical proof of this fact, as was done in [13] for the Haldane–Shastry chain, since the equivalent form of the Hamiltonian $H - \epsilon_{N/2}$, in contrast to [13], is no longer evidently positive everywhere except for this state.

For odd $N = 2L + 1$, the states with minimal total spin and its projection are given by $M = L$. The Q-polynomials, as follows from (18), should have degree 1 in each variable $\{z_j\}$, being symmetric in $\{z_j\}$. This means that Q should be linear in the variables

$$a_1 = \prod_{j=1}^{L} z_j \qquad a_l = [(l-1)!]^{-1}\hat{D}^{l-1}a_1 \qquad 2 \leqslant l \leqslant L$$

where $\hat{D} = \sum_{j=1}^{L}\partial/\partial z_j$. Equation (18) transforms into a system of $(L+1)$ linear equations for the corresponding coefficients which gives the set of only $L+1$ energies of the spin chain. A similar reduction procedure works in each M-magnon sector where the degree of the Q-polynomial in each variable $\{a_l\}$ should be $N - 2M$.

In conclusion, we have demonstrated that the quantum spin chains defined on a non-uniform lattice given by the roots of the equilibrium equations may be completely integrable by the construction of the explicit form of the Lax relation and one of the conserved currents. We have calculated some of their eigenenergies for the special case in which the solutions to (8) are expressed in terms of the roots of some polynomials. This condition leads to the restriction (12) on the parameters of an external potential confining the classical Sutherland systems.

The construction of the whole set of eigenvectors and finding other conserved currents of the model remains an interesting open problem. It might probably be solved within the framework of the general scheme of the quantum inverse scattering method adapted in [14] to systems with non-trivial boundary conditions. However, at present there is no explicit way of applying this scheme to our model.

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